Optimizing Counter-Terror Operations

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Optimal Control of Terroristic Attacks

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Roadmap

• Introduction
• Two Sides of a Coin
• Terminology of Differential Games
• Obama vs. Osama
• Leader – Follower Games
• A Post September 11th Game on Terrorism
• Fire & Water Strategies
• Public Opinion as ‘Catalysator’
• Conclusions
Introduction

Modeling a “Stock” of Terrorists is not Common but Has Precedent

• Keohane and Zeckhauser, 2003: „The ecology of terror defense“, state = ITO’s resources
• Castillo-Chavez & Song, 2003: „Models for the transmission dynamics of fanatical behaviors“
• Faria, 2004: „Terror cycles“
• Kaplan et al., 2005: „What happened to suicide bombings in Israel? Insights from a terror stock model“
• Udwadia et al., 2006: „A dynamical model of terrorism“
• Caulkins et al., 2007: fire and water strategies for counter-terrorism, “double-edged sword effect”; terrorism and global reputation
• Grass et al., 2008
• Kress & Szechtman, 2009: effect of intelligence in counter-insurgency operations
• Think of there being a population or “stock” of terrorists

• **Focus on “flows” in and out of that stock**
  – Pay attention to the possibility that some interventions that increase an *outflow* might simultaneously increase *inflow*
  – Levels of associated “reputation” or “perception” stocks can affect those flows
Data Limitations

• Terror stock data are poor
  – So models are stylized and not validated
  – Value stems from articulating ideas in a language as precise as mathematics
    • Not from hypothesis testing
    • Providing specific, quantitative guidance
Two Sides of a Coin
A simple counter-terror model

\[ x(t) \ldots \text{number of terrorists at time } t \]
\[ y(t) \ldots \text{power of the government} \]
\[ u(t) \ldots \text{counter-terror measures} \]

\[
\begin{align*}
\dot{y} &= u - \delta y \\
\dot{x} &= g(x) - \alpha y
\end{align*}
\]

\[ \delta \ldots \text{obsolence rate} \]
\[ g(x) \ldots \text{growth function} \]
\[ \alpha > 0 \ldots \text{impact of accumulated terrorist fighting} \]
government wants to minimize both the damage of terroristic attacks and the cost of its measures:

\[
\min_{u \geq 0} \int_0^\infty e^{-rt} [D(x) + C(u)] \, dt
\]

\(D(x)\) ... damage, \(C(u)\) ... costs
4-D canonical system

Supercritical Hopf-bifurcation: stable limit cycles, persistent oscillations make economic sense

Wirl & Feichtinger (1992)
payoff rate $g^i(x, u^1, ..., u^N, t), \quad i = 1, \ldots, N$

$$V^i(u^1, ..., u^N) = \int_0^T e^{-r_i t} g^i(x, u^1, ..., u^N, t) \, dt + e^{-r_i T} S^i(x(T))$$

infinite-horizon problem

$$V^i(u^1, ..., u^N) = \int_0^\infty e^{-r_i t} g^i(x, u^1, ..., u^N, t) \, dt$$
Markovian (Feedback or Closed-loop) Information Pattern

\[ u^i(t) = \varphi^i(x(t), t) \]

stationary Markovian strategy

\[ u^i(t) = \varphi^i(x(t)) \]

Open-loop Information Pattern

\[ u^i(t) = \varphi^i(t) \]
Tractable Game Structures

• Linear quadratic games
  Engwerda (2005)

• State-separable games (Linear state games)
  Dockner et al. (1985), Dockner et al. (2000, Sect. 7.2)

\[ \mathcal{H}_{xx}^i = 0 \quad \text{and} \quad \mathcal{H}_{ux}^i = 0 \]

for \[ \mathcal{H}_{ui}^i = 0 \]

Open-loop Nash equilibria qualify as Markovian solution
Obama vs. Osama

A dynamic game of the government vs. ITO

\[ x(t) \ldots \text{resources of the terrorists at time } t \]
\[ v(t) \ldots \text{intensity of attacks; strategic} \]
\[ \quad \text{instrument of the terrorists} \]
\[ u(t) \ldots \text{counter-terror measures} \]

Player 1: Obama (US): resource exploiter
Player 2: Osama (ITO): resource conservator
\[
\dot{x} = g(x) - h(u, v), \quad x(0) = x_0 > 0
\]
\[
x(t) \geq 0 \quad \forall \quad t \geq 0
\]

g(x) \quad \ldots \text{growth function (hump-shaped)}

h(u,v) \quad \ldots \text{‘harvesting’ function}

\[
h_u, h_v > 0, h_{uu} \leq 0, h_{vv} \geq 0, h_{uv} \geq 0
\]
1st player: government: resource exploiter

\[
I_1 = \int_0^\infty e^{-\rho_1 t} [h(u, v) - f(x) - k(v) - a(u)] \, dt \rightarrow \max_u
\]

\[f > 0, f' > 0; k > 0, k' > 0, k'' \geq 0;\]
\[a > 0, a' > 0, a'' \geq 0\]

2nd player: ITO: resource conserver

\[
I_2 = \int_0^\infty e^{-\rho_2 t} [\sigma(x) + \beta(v)] \, dt \rightarrow \max_v
\]

\[\sigma > 0, \sigma' > 0, \sigma'' \leq 0; \beta > 0, \beta'' \leq 0\]
\[ H_1 = h(u,v) - f(x) - k(v) - a(u) + \lambda [g(x) - h(u,v)] \]
\[ H_2 = \sigma(x) + \beta(v) + \mu [g(x) - h(u,v)] \]

\[ \lambda, \mu \] shadow prices of the stock \( x \) to player 1 and 2

\[ \frac{\partial H_1}{\partial u} = (1 - \lambda)h_u(u,v) - a'(u) = 0 \quad \rightarrow \quad u^* = u^*(\lambda, v) \]
\[ \frac{\partial H_2}{\partial v} = \beta'(v) - \mu h_v(u,v) = 0 \quad \rightarrow \quad v^* = v^*(\mu, u) \]
Proposition 1.

(i) The Hamiltonian of player 1 is concave in the control $u$ as
\[ \lambda < 1. \]

(ii) The shadow price $\mu$ is positive and $H_2$ is thus concave in $v$.

(iii) The shadow price $\lambda$ is positive, iff the marginal utility $h_u$ of the governments' activities exceeds the marginal costs $a'(u)$.
Proposition 2. Given the specifications of functions as above, a necessary condition for the existence of purely imaginary eigenvalues is that the terrorists are more impatient than the governments, i.e.\
\[ \rho_2 > \rho_1 \] has to hold.
\[ \rho_1 = 0.29, \quad \rho_2 = 0.49, \quad r = 0.63, \quad \pi = 0.97, \]
\[ \sigma = 0.55, \quad f = 0.43, \quad b = 0.79, \quad \eta = 0.11. \]

\[ a_{\text{crit}} = 2.128539 \]

computer code BIFDD

\[ A = -0.08649, \quad D = -0.00822 \]

Hopf bifurcation at \( a = a_{\text{crit}} \)

Stable cycles at \( a < a_{\text{crit}} \)
Periodic solutions in the 3-dimensional state-control space. The unstable cycle (dashed-dotted line) intersects the $x = 0.3$ plane at points $A$ and $B$. Orbits starting on the dashed curve (e.g. in points 1, 2, 3) converge against the stable cycle.
Time paths of the stock and the controls along one period of the cyclical solution
A Post September 11\textsuperscript{th} Game on Terrorism

\begin{align*}
\dot{x} &= \rho x, \quad x(0) > 0 \\
\dot{x} &= \rho x - \varphi u + \frac{\alpha}{2} u^2 - vv - \beta uv, \quad x(0) > 0
\end{align*}

$x \geq 0$ number of terrorists (state variable),
$u \geq 0$ intensity of the target country’s terror-control activities, relative to its maximum sustainable level (control variable of the “target country”),
$v \geq 0$ number of ITO attacks (control variable of ITO),
$\rho \geq 0$ endogenous growth rate of the ITO,
$\varphi \geq 0$ rate at which terror-control operations would deplete the ITO if the target country were on a full counter-offensive,
\[ \frac{\alpha}{2} \geq 0 \] growth rate of the ITO due to collateral damage induced by the target country’s terror-control attacks \( u \),

\[ \beta \geq 0 \] ratio of ITO operatives lost per successful attack when the target country is on full counter-offensive, \( (u = 1) \) vs. none at all,

\[ \nu \geq 0 \] average number of terrorists killed or captured per attack.

\[
\min_{u(\cdot)} \int_0^\infty e^{-r_1t}(c_1x + c_2\nu + c_3u)dt \\
\max_{\nu(\cdot)} \int_0^\infty e^{-r_2t}(b_1x + b_2\nu + b_3u - \frac{c_4}{2}\nu^2)dt
\]

\[ x, u, \nu \geq 0 \]
Nash equilibrium

\[ \mathcal{H}^1 = -c_1 x - c_2 v - c_3 u + \lambda \left( \rho x - \left( \varphi + \frac{\alpha}{2} \right) u - vv - \beta uv \right) \]

\[ \mathcal{H}^2 = b_1 x + b_2 v + b_3 u - \frac{c_4}{2} v^2 + \mu \left( \rho x - \left( \varphi + \frac{\alpha}{2} \right) u - vv - \beta uv \right) \]

\[ \dot{\lambda} = (r^1 - \rho) \lambda + c_1 \]

\[ \dot{\lambda} = -\frac{c_1}{r^1 - \rho} \]

\[ \dot{\mu} = (r^2 - \rho) \mu - b_1 \]

\[ \dot{\mu} = \frac{b_1}{r^2 - \rho} \]
\( H_u^1 = 0 \) implies \( u^* = \frac{1}{\alpha} \left( \frac{c_3}{\lambda} + \varphi + \beta v \right) \)

\( H_v^2 = 0 \) implies \( v^* = \frac{b_2}{c_4} - \frac{\mu}{c_4} \left( v + \beta u \right) \)

Stationary Nash equilibrium

\[
\begin{align*}
\hat{u}_N &= \frac{\beta (b_2 - \hat{\mu} v) + c_4 (\varphi + c_3/\hat{\lambda})}{c_4 \alpha + \hat{\mu} \beta^2} \\
\hat{v}_N &= \frac{\alpha (b_2 - \hat{\mu} v) - \hat{\mu} \beta (\varphi + c_3/\hat{\lambda})}{c_4 \alpha + \hat{\mu} \beta^2} \\
\hat{x}_N &= \frac{1}{\rho} \left( \varphi - \frac{\alpha}{2} \hat{u}_N \right) \hat{u}_N + (v + \beta \hat{u}_N) \hat{v}_N
\end{align*}
\]

\( \hat{x}_N \) unstable steady state
Table 1. Comparative static analysis of the interior stationary Nash equilibrium $(\hat{u}_N, \hat{v}_N)$ of the terror game (8.36–8.38) with respect to the model parameters for $c_3 = 0$. + means that $\partial \hat{u}_N / \partial \text{parameter} > 0$, – the opposite case, 0 means that the parameter has no influence on the control, and ? denotes ambiguity.

<table>
<thead>
<tr>
<th>parameter / Nash equilibrium</th>
<th>$\rho$</th>
<th>$\varphi$</th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$\nu$</th>
<th>$c_1$</th>
<th>$c_2$</th>
<th>$b_1$</th>
<th>$b_2$</th>
<th>$b_3$</th>
<th>$c_4$</th>
<th>$r^1$</th>
<th>$r^2$</th>
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<tbody>
<tr>
<td>$\hat{u}_N$</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>?</td>
<td>-</td>
<td>0</td>
<td>0</td>
<td>-</td>
<td>+</td>
<td>0</td>
<td>?</td>
<td>0</td>
<td>+</td>
</tr>
<tr>
<td>$\hat{v}_N$</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>?</td>
<td>-</td>
<td>0</td>
<td>0</td>
<td>-</td>
<td>+</td>
<td>0</td>
<td>?</td>
<td>0</td>
<td>+</td>
</tr>
<tr>
<td>$\hat{x}_N$</td>
<td>?</td>
<td>?</td>
<td>?</td>
<td>?</td>
<td>?</td>
<td>0</td>
<td>0</td>
<td>-</td>
<td>+</td>
<td>0</td>
<td>?</td>
<td>0</td>
<td>+</td>
</tr>
</tbody>
</table>
The Stackelberg Equilibrium with the ITO as Leader

**Step 1:** ITO announces \( v \)

**Step 2:** \( u^* = u^*(v) = \frac{1}{\alpha} (\varphi + \beta v) \) \hfill (**)

**Step 3:** \( \max_{v(\cdot)} \int_0^\infty e^{-r^2t} \left( b_1x + \left( b_2 - \frac{c_4}{2} v \right)v + b_3 u^*(v) \right) dt \)

s.t. \( \dot{x} = \rho x - \left( \varphi + \frac{\alpha}{2} u^*(v) \right) u^*(v) - vv - \beta u^*(v)v, \)

\( \dot{\lambda} = (r^1 - \rho) \lambda + c_1 \)

\( u^*(v) \) given by \hfill (**)

Due to that state separability the adjoint variable $\psi$ of the leader w.r.t. the adj. var. $\lambda$ of the followers plays no role in the opt. problem.

\[
\mathcal{H}^2 = b_1 x + \left( b_2 - \frac{c_4}{2} \nu \right) \nu + b_3 u^*(\nu) + \mu \dot{x} + \psi \dot{\lambda}
\]

\[
\hat{u}_S = \frac{\beta (b_2 - \hat{\mu} \nu) + c_4 \varphi + c_3 (\beta^2 / \alpha)}{c_4 \alpha + \hat{\mu} \beta^2}
\]

\[
\hat{\nu}_S = \frac{\alpha (b_2 - \hat{\mu} \nu) - \beta (\hat{\mu} \varphi - b_3)}{c_4 \alpha + \hat{\mu} \beta^2}
\]

\[
\hat{x}_S = \frac{1}{\rho} \left( \left( \varphi - \frac{\alpha}{2} \hat{u}_S \right) \hat{u}_S + (\nu + \beta \hat{u}_S) \hat{\nu}_S \right)
\]
Comparison of the Nash and Stackelberg Equilibrium Solutions

\[ \hat{u}_S = \hat{u}_N + \frac{\beta}{\alpha} \Delta, \quad \hat{v}_S = \hat{v}_N + \Delta \]

\[ \Delta := \frac{b_3 \beta}{c_4 \alpha + \mu \beta^2} \]

\[ \hat{u}_S > \hat{u}_N \quad \text{and} \quad \hat{v}_S > \hat{v}_N \]

\[ \hat{x}_S - \hat{x}_N > 0 \]

\[ V_S^2 - V_N^2 = \frac{b_3 \beta \Delta}{2 \alpha r^2} > 0 \]
Fire & Water Strategies

„Fire strategies“: territorial bombing, aggressively searching all people, activities involving significant collateral damage
→ inconvenience to third parties, resentment by population, stimulation of recruitment rates, elimination of current terrorists

„Water strategies“: intelligence driven arrests or „surgical“ operations against almost certainly guilty individuals
→ no harm to innocent parties, higher acceptance by population, expensive, difficult to apply
Occurrence of terroristic attack at $t=0$

Stage 1: modest counter measures, „water strategy“

Stage 2: additional, more aggressive measures, „fire strategy“
   (side effect: increased inflow of recruits to terror organisation)

$x(t)$ … number of terrorists at time $t$
$u(t)$ … „water strategy“ at time $t$
$v(t)$ … „fire strategy“ at time $t$
Stage 1:

\[ \int_0^{t_s} e^{-rt} (cx + u^2) \, dt \]

s.t. \[ \dot{x} = \tau + kx^\alpha - \mu x - \beta ux^\theta, \quad x(0) = x_0 \]

Stage 2:

\[ \int_{t_s}^\infty e^{-rt} (cx + u^2 + v^2) \, dt \]

s.t. \[ \dot{x} = \tau + k(1 + pv)x^\alpha - \mu x - \beta ux^\theta - \left( \frac{\gamma_0 - \varepsilon}{1 + t_s} \right) vx \]

where \[ \gamma_0 > \varepsilon \]
<table>
<thead>
<tr>
<th>Variables</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r$</td>
<td>discount rate</td>
<td>0.05</td>
</tr>
<tr>
<td>$c$</td>
<td>costs per unit of terrorists</td>
<td>10</td>
</tr>
<tr>
<td>$\tau$</td>
<td>constant inflow</td>
<td>0</td>
</tr>
<tr>
<td>$\rho$</td>
<td>contribution of fire control to recruitment</td>
<td>0.2</td>
</tr>
<tr>
<td>$k$</td>
<td>normalization factor</td>
<td>0.05</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>influence of actual state on recruitment</td>
<td>0.5</td>
</tr>
<tr>
<td>$\mu$</td>
<td>“natural” per capita outflow</td>
<td>0.05</td>
</tr>
<tr>
<td>$\beta$</td>
<td>efficiency of water operations</td>
<td>0.01</td>
</tr>
<tr>
<td>$\theta$</td>
<td>diminishing returns from water operations</td>
<td>0.5</td>
</tr>
<tr>
<td>$\gamma_0$</td>
<td>maximum efficiency of fire operations</td>
<td>0.1</td>
</tr>
<tr>
<td>$\varepsilon$</td>
<td>minimal efficiency</td>
<td>0</td>
</tr>
</tbody>
</table>

Parameters for the two-stage fire-and-water model.
Panel (a) depicts the optimal path in the state–costate space for $x_0 = 0.2$. The number of terrorists grows during the first stage peaks at the switching time $t_s = 8.3135$ and converges to $(\hat{x}, \hat{\lambda})$. In panel (b) the corresponding optimal controls $u(\cdot)$ (solid curve) and $v(\cdot)$ (dashed curve) are shown.
Public Opinion as Catalysator

• Is it important to manage public opinion while fighting terrorism?

Caulkins et al. (2007):

x … size of the terrorist organization
y … reputation, level of public sympathy for counter-terror forces
u … counter-terror operations

y decreases with u but increases with x
\[
\min_{u(\cdot)} \int_0^\infty e^{-rt} \left( cx + \frac{u^2}{2} \right) dt
\]

s.t. \[
\dot{x} = \tau + kx^\alpha - \mu x - \eta u x^\beta y^\gamma \\
\dot{y} = \delta x - \rho u^2 + \kappa(b - y) \\
x(0) = x_0 \geq 0, \quad x(0) = x_0 \geq 0 \\
u \geq 0
\]
\[ H = \lambda_0 \left( cx + \frac{u^2}{2} \right) + \lambda_1 \left( \tau + kx^\alpha - \mu x - \eta ux^\beta y^\gamma \right) \\
+ \lambda_2 \left( \delta x - \rho u^2 + \kappa (b - y) \right) \]

\[ H_u = u - \lambda_1 \eta x^\beta y^\gamma - 2\rho \lambda_2 u = 0 \]

\[ H_{uu} = 1 - 2\rho \lambda_2 \]

\[ u^* = \frac{\lambda_1 \eta x^\beta y^\gamma}{1 - 2\rho \lambda_2} \]
Panel (a) depicts the neighborhood of the equilibria, with the one-dimensional stable manifold of $\hat{E}_m$ (gray line) and two solution paths (dark line). The dashed line shows the line along which the solution is continued from $\hat{E}_h$ to $\hat{E}_l$ and *vice versa*. 
In panel (a) four optimal paths (black solid curves), denoted by 1, 2, 3, 4, are depicted. Path 1 and 2 converge to the high equilibrium $E_h$ and the paths 3 and 4 move to the low equilibrium $E_l$. The gray solid line is a weak DNSS curve approaching the medium equilibrium $E_m$. The dashed black curve is a DNSS curve and the paths 2 and 3 are two equally optimal solutions starting at one of DNSS points. The weak DNSS curve and DNSS curve meet at point $\otimes$, which is the last weak DNSS point. In panel (b) the trajectories 1–4 and weak DNSS curve are depicted in the neighborhood of the equilibria.
Conclusion

• Charlatanerie in violence / peace „research“

• Stock-flow models of counter-terrorism

• Qualitative insights: advantage of the maximum principle

• Persistent oscillations (stable limit cycles) as (open-loop) Nash solutions

• Complex behavior of „optimal“ strategies in differential games

• Statement of J. Lesourne
Optimal Control and Differential Games of Violence
How to Fight Terror and Insurgencies

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Joint work with A.J. Novak, A.Seidl, G. Tragler, and S. Wrzaczek
Outline

1. Research Goals and Methodological Objectives
2. Optimizing Counter-terror Operations in Asymmetric Lanchester Models
3. Multi-Stage Differential Games
4. Popular Support to Terror and Counter-Terror
Introduction

- **OR & ManSci**: Modeling of dynamic optimization problems both with and without strategic interactions
- Seierstad & Sydsaeter (1987); Leonard & Long (1992); Sethi & Thompson (2000); Feichtinger & Hartl (1986); Grass et al. (2008)
- Basar & Olsder (1995); Dockner et al. (2000); Long (2010)
- Future work: combining multi-stage optimal control and optimal impulse control methods with dynamic games
- War on Terror: terror attacks & counter-terror measures
- enormous economic problems, but only very few ‘hard-core’ dynamic optimization models
- MAtematics of PUBlic SEcurity (MAPUSE)
- Conflicts between governments and terrorists: both continuous policy changes and abrupt ‘game-changing’ events
Research Goals

1. Mathematical advances - both analytical and numerical
2. Modeling war on terror

Methodological issues

(i) *Bifurcation theory of nonlinear dynamical systems*: multiple equilibria and limit cycles; mechanisms generating complex behavior

(ii) *Multi-stage modeling*: switch between consecutives regimes; timing of switch: exogenous, stochastic or endogenous.

(iii) *Differential games*: interactions between different decision makers in a dynamic framework

(iv) *Impulse control*: control instruments executed only at optimally determined points in time (not continuously!)
Optimizing Counter-terror Operations in Asymmetric Lanchester Models

- Lanchester (1916): Battle between two parties
- Guerilla warfare: Deitchman (1962), Schaffer (1968)
- Kress & Szechtman (2009), Kaplan et al. (2010): levels of intelligence, government vs. small guerrilla group
- Asymmetric nature of counter-terror
  - Government forces: limited awareness of enemies
  - Guerillas: can observe location of government forces
- Lanchester square law, Lanchester linear law
Kress & Szechtman (KKS): Lanchester paradigm with intelligence

2 state variables:
- $G$ ... size of government forces
- $I$ ... size of terrorist group ($0 \leq I \leq 1$)

intelligence rate $\mu \in [0, 1]$

attrition rates $\alpha$ and $\gamma$

$\beta$ & $\epsilon$ recruitment rates of government and terrorists

$$
\dot{G} = -\alpha I + \beta \\
\dot{I} = -\gamma G (\mu + (1 - \mu)I) + \epsilon + \theta(C)
$$
Double edge sword effect $\theta(C)$
Collateral casualities

$$C = \gamma G (1 - \mu)(1 - I)$$

$\mu = 1$: classical Lanchester’s square law
$\mu = 0$: guerrilla model
Control variables:
- intelligence rate $\mu$
- recruitment rate $\alpha$
Lanchester-type Models

**Resulting Optimization Problem**

\[
\min_{\mu, \beta} \int_0^\infty e^{-rt}\left(D(I) + C(\mu) - A(G) + K(\beta)\right) \, dt,
\]

\[s.t. \quad \dot{G} = -\alpha I + \beta,\]

\[\dot{I} = -\gamma G(\mu + (1 - \mu)I) + \epsilon + \theta(C),\]

\[0 \leq \mu \leq 1, \quad G \geq 0, \quad I \geq 0.\]

- \(D(I)\) ... damage created by insurgents
- \(C(\mu)\) ... cost of intelligence
- \(A(G)\) ... costs of army
- \(K(\beta)\) ... costs of hiring and firing
Analysis of a Dynamic Version of the Lanchester Model

- Since then: *descriptive* models
- Optimal control models: Multiple equilibria and path-dependence:
  - occur in several models of counter-terror (e.g., Caulkins et al., 2008, 2009)
  - “tipping” between two or more long-run solutions
  - “eradicate” vs. “accomodate” policy
  - Existence of three-fold indifference-threshold point in the \((G, I)\)–state space?
  - Conjecture by F. Wagener: OC models with \(n\) state variables, multiplicity of indifference points at most \(n + 1\)
- Persistent oscillations
Lanchester Model in a Differential Game Setting

- *Lucas critique* applies for terror OC models
- Feedback mechanisms within Lanchester model in a DG
Multi-Stage Differential Games Between a Government and a Terrorist Organization

- DG between a government (player 1) and ITO (player 2)
- multi-stage approach in OC: see e.g. Tomiyama & Rossana (1989), Makris (2001), Grass et al. (2008, Sect. 8.1.3)
- state $x$: number of terrorists
- control instruments of government:
  - $u_1$: water measures: surgical operations
  - $v_1$: fire measures: also kill civilians; stimulate recruitment rates of terrorists
- control instruments of terrorists:
  - $u_2$: attack rate
- P1: disutility from $x$, $u_1$, $v_1$ (if used) and $u_2$
- P2: utility and disutility from same quantities, costs from $u_2$
- ITO’s political objectives: positive assessment of both strategies of P1
Applying Multi-stage Methods and Impulse Controls to Differential Games

**Stage 1:** Substantial terror attack at $t = 0$; P1 applies $u_1$, P2 uses $u_2$

**Stage 2:** At time $t_s$, P1 decides to switch from water to fire; P1 uses $v_1$ and $u_1$, P2 attacks with $u_2$


- Two or more decision makers influence switching time between consecutive regimes

- *Anticipation of Terror Attacks and Counter Measures*
  - Government expects terror attack
  - Government cannot anticipate terror attack
  - Government knows when attack will happen (but cannot prevent it)
  - Government tries to prevent (impact) of terror attack
Popular Support to Terror and Counter-Terror

- Fighting terrorism means influencing public opinion
- Civilian population between the terrorists and government
- Both players need popular support
- Population source for recruitment of terrorists
Caulkins et al. (2009) introduced reputation (level of public sympathy for counter-terror forces) as second state variable.

Fighting terrorists: higher security for population, but collateral damage to innocent parties (➔ erodes sympathy for government).

OC model, government decision maker, who minimizes strength of terrorists and costs for counter-terror operations.

See also Grass et al. (2008, Sect. 6.3)

Existence of multiple equilibria, indifference curve.
- **Extension**: terror organisation as additional decision maker
- two-person non-zero sum DG
- *Catalysator*-type property of reputation: if reputation is high, counter-terror measures more effective
- Catalysator effect: general character of variety of problems: e.g., contagious diseases (quarantines and shutting down of public transport)
Fighting Terror in a Host Country

- Population’s interaction with terrorist as **three player DG**: 
  - Population as additional player
  - Brito & Intriligator (1992): guerrillas, government and drug lords
  - Langlois & Langlois (2011): repeated game between US, host country and terrorists
    - host will push back against terrorists in anticipation of future involvement of Western power

- More than three players: local insurgents with local objectives

- **Multi-player Games**: Two parties might cooperate
Terror Escalations Along the Progress Curve

- Johnson et al. (2011):
  - number of attacks as proxy for accumulated terrorist experience
  - timing of terror attacks power progress law
  - escalations of terror shrinks intervals of attacks
  - escalation rate (speed of terrorist learning) between 0 and 1.5

- Hartl et al. (2012): impulse optimal control model